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CHORDS AND DISJOINT PATHS IN MATROIDS*

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A chord of a circuit C of a matroid M on E is a cell $e \in E \setminus C$ such that C spans e. Menger's theorem gives necessary and sufficient conditions for a cell of a graphic matroid to be a chord of some circuit. We extend this result to a large class of matroids and find all minimal counterexamples. The theorem is used to obtain results on disjoint paths and to characterize a class of matroid sums.

1. Introduction

A *chord* of a circuit C of a matroid M on E is a cell $e \in E \setminus C$ such that C spans e. We ask the question: Given a cell e of M, when is e a chord of a circuit of M. We consider the case where M is non-separable, since only circuits in the same component as e can span e; we also assume that M has at least 3 cells.

If M is graphic (that is, its circuits are the edge-sets of polygons of a finite undirected graph G) e is a chord of a circuit if and only if there exist in $G - e$ (the graph obtained from G by deleting e) two internally vertex-disjoint paths joining the ends u and v of e. If G is non-separable and has at least three edges, it is obvious from Menger's theorem that such paths exist if and only if $G - e$ is non-separable. In view of a standard result on separability in graphic matroids, the following result is just a special case of Menger's theorem.

Theorem 1.1. *Let e be a cell of a non-separable graphic matroid M having at least 3 cells. Then e is a chord of a circuit of M if and only if $M \setminus e$ is non-separable.*

The main result of this paper is a generalization of Theorem 1.1, but in a direction different from the one in which Menger's theorem generalizes Theorem 1.1; namely, we prove an extension of Theorem 1.1 to a larger class of matroids. The matroids $F_1, F_2, \ldots, F_5$ are obtained from Fig. 1 in the following way: the cells are the points, the circuits of cardinality less than 5 are the complements of the lines, and the remaining circuits are the subsets of cardinality 5 not containing the complement of any line. Where e is as indicated in Fig. 1, for each i, F_i and $F_i \setminus e$ are non-separable matroids, but e is a chord of no circuit.

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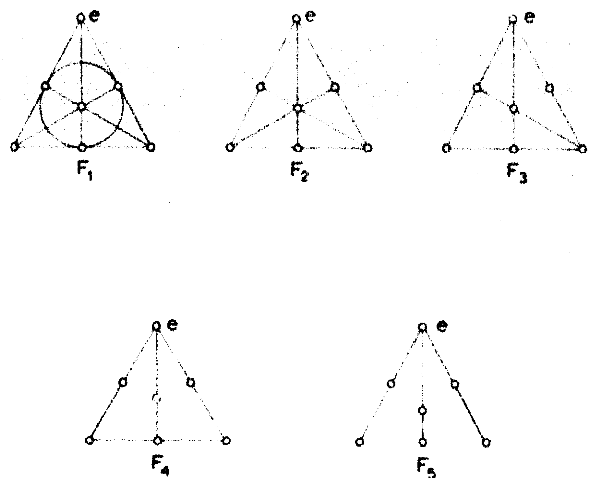

Fig. 1.

On the other hand, we can exclude these counterexamples by considering M and e satisfying the following condition.

(1) There is no isomorphism i from a minor of M to any of $F_1, \ldots, F_5$ such that $i(e) = e$.

Theorem 1.2. *Let e be a cell of a non-separable matroid M having at least 3 cells and suppose M and e satisfy* (1). *Then e is a chord of a circuit of M if and only if $M \setminus e$ is non-separable.*

Specializations of Theorem 1.2 are obtained by observing that the only binary F_i is F_1 and that no F_i is regular. Since every graphic matroid is regular, Theorem 1.2 does extend Theorem 1.1.

A variant of the problem considered above is: when does there exist a set $A \subseteq E - e$ such that A spans e and each minimal non-empty separator of $M \setminus (E \setminus A)$ is a circuit. In the case of graphic matroids this reduces to a question of the existence of edge-disjoint paths, and the answer is provided by the max-flow min-cut theorem: it is necessary and sufficient that e be contained in no cocircuit of cardinality 2 or less. Using Theorem 1.2 we will show that this result also extends to M and e satisfying (1). When M is binary this result gives a theorem on what could be called "disjoint paths"; this result is a special case of a theorem of Seymour. We will consider some other problems related to disjoint paths.

Another quite different application of Theorem 1.2 is a characterization of those binary matroids, having no minor isomorphic to F_1, which are non-trivial matroid sums. This result extends a theorem of Lovász and Recski on graphic matroids.

2. Preliminaries

We abbreviate $A \cup \{e\}$ to $A + e$ and $A \setminus \{e\}$ to $A - e$. We will use elementary matroid terminology as it appears in the more-or-less standard references [5, 8, 15].

The papers of Tutte [13, 14] are also important references, but much of his terminology, as in his notion of rank, is "dual" to ours. The reader is expected to be familiar with the notion of separability and its characterization in terms of circuits, rank and existence of direct sums. We denote by M^* the matroid dual to M; the circuits of M^* are called *cocircuits* of M.

A set A is said to *span* cell e of M if, where r is the rank function of M, $r(A+e)=r(A)$. Let $A \subseteq E$ and M be a matroid on E. We use $M \setminus A$ to denote the matroid on $E \setminus A$ obtained by *deleting* A from M; its circuits are the circuits of M contained in $E \setminus A$. We use M/A to denote the matroid on $E \setminus A$ obtained by *contracting* A from M, defined to be $(M^* \setminus A)^*$; its circuits are minimal sets C such that $C \cup A$ contains an M-circuit not contained in A. We abbreviate $M \setminus \{e\}$ to $M \setminus e$ and $M/\{e\}$ to M/e. A *minor* of M is a matroid of the form $(M \setminus A)/B$ $(=(M/B)\setminus A)$, where A and B are disjoint. A cell e of M is a *loop* if $\{e\}$ is a circuit, and is an *isthmus* if $\{e\}$ is a cocircuit.

A matroid is *binary* if it is isomorphic to the linear independence matroid of a matrix over GF(2). Minors of binary matroids are binary, and the symmetric difference of any number of circuits in a binary matroid is a union of disjoint circuits. A matroid is *regular* if it is isomorphic to the linear independence matroid of a totally unimodular matrix over the real field. Regular matroids are binary, and their minors are regular; the matroid F_7 and its dual are binary but not regular.

The *sum* [5] of matroids M_1 on E_1 and M_2 on E_2 is the matroid $M_1 + M_2$ on $E_1 \cup E_2$ whose independent sets are sets of the form $I_1 \cup I_2$ where I_i is M_i-independent, $i = 1$ and 2. A partition $\{E_1, E_2\}$ of E is called [14] a *2-separation* of M on E if $|E_1|, |E_2| \geq 2$ and $r(E_1) + r(E_2) \leq r(E) + 1$. A matroid M is *3-connected* if it is non-separable and has no 2-separation.

Finally, we make use of a matroid decomposition which is a special case of a theory due to Jack Edmonds and the author which is described in [4]. This material is closely related to a theory due to Bixby [1]. Let $\{E_1, E_2\}$ be a partition of E and let $f \notin E$. Where M_i is a matroid on $E_i + f$ for $i = 1$ and 2, we define $M_1 * M_2$ to be $(M_1 + M_2)/f$. The following results appear in [4].

(2) Where $\mathscr{C}(\cdot)$ denotes the set of circuits of a matroid, $\mathscr{C}(M_1 * M_2) = \mathscr{C}(M_1 \setminus f) \cup \mathscr{C}(M_2 \setminus f) \cup \{(C_1 \cup C_2) - f : f \in C_i \in \mathscr{C}(M_i),\ i = 1 \text{ and } 2\}$. It follows that, if f is not a loop or an isthmus in M_1 or M_2, then $M_1 * M_2$ is non-separable if and only if M_1 and M_2 are.

(3) If M is non-separable and $\{E_1, E_2\}$ is a 2-separation of M, then there exist M_1 and M_2 as above such that $M = M_1 * M_2$.

(4) If $M = M_1 * M_2$ and $A \subseteq E_1$, then $(M \setminus A) = (M_1 \setminus A) * M_2$ and $M/A = (M_1/A) * M_2$. We can deduce from this and (2) that if $M_1 * M_2$ is non-separable and $e \in E_1$, then there exists a minor M' of $M_1 * M_2$ on $E_2 + e$ such that M_2 can be obtained from M' by replacing e by f.

3. Chords of circuits

The first part of this section is devoted to a proof of Theorem 1.2. It is easy to check that Theorem 1.2 is true for matroids having exactly 3 cells. Throughout the proof M will denote a non-separable matroid on E having a cell e such that $M\backslash e$ is non-separable, but having no circuit of which e is a chord. We also choose M so that no proper minor of M has all of these properties (with respect to the same element e); briefly, M is chosen to be "minimal". We will show that M is isomorphic to one of the matroids F_i, and that the isomorphism is the identity on e.

Lemma 3.1. *M is 3-connected.*

Proof. If M is not 3-connected there exist by (3) non-separable matroids M_i on E_i+f, $i=1$ and 2, such that $M=M_1*M_2$. We may assume $e\in E_1$. Then M_1 is isomorphic to a proper minor of M by (4), and $M_1\backslash e$ is non-separable by (4) and (2). Therefore, by the minimality of M, e is a chord of a circuit C_1 of M_1. If $f\notin C_1$, we take $C=C_1$; if $f\in C_1$, we take $C=(C_1\cup C_2)-f$, where C_2 is an M_2-circuit containing f. In either case C is an M-circuit and it is easy to see that it M-spans e, since C_1 M_1-spans e. This is a contradiction; it follows that M must be 3-connected.

Lemma 3.2. *For every $f\in E-e$ there exists a circuit $C\subseteq E\backslash\{e,f\}$ such that $C+f$ M-spans e.*

Proof. It follows from the minimality of M that $M\backslash\{e,f\}$ is separable. Since $M\backslash e$ is non-separable, $(M\backslash e)/f$ is non-separable by a well-known result of Tutte [14]. Thus by the minimality of M, e is a chord of a circuit C of M/f. Now $C+f$ M-spans e and so cannot be a circuit of M. Thus C must be a circuit of M.

Lemma 3.3. *For every $f\in E-e$ there exists g such that $\{e,f,g\}$ is a circuit of M^*.*

Proof. By the minimality of M, $M\backslash\{e,f\}$ is separable, so there exists a non-trivial partition $\{E_1,E_2\}$ of $E\backslash\{e,f\}$ such that $r(E_1)+r(E_2)=r(E\backslash\{e,f\})$. Since M has at least 4 cells and is 3-connected, $r(E\backslash\{e,f\})=r(E)$. We may assume the circuit C of Lemma 3.2 is contained in E_1. Then it follows from Lemma 3.2 that $r(E_1\cup\{e,f\})\leq r(E_1)+1$, so $r(E_1\cup\{e,f\})+r(E_2)\leq r(E)+1$. Since M is 3-connected, it must be that $|E_2|=1$, say $E_2=\{g\}$; $\{g\}$ is a separator of $M\backslash\{e,f\}$, so g is an isthmus of $M\backslash\{e,f\}$. Thus $\{e,f,g\}$ is a circuit of M^* (since no proper subset can be, by 3-connectivity).

Lemma 3.4. *$E=\{e,f_1,g_1,f_2,g_2,f_3,g_3\}$ where*
(a) *$\{e,f_i,g_i\}$ is an M-cocircuit for each i;*
(b) *$\{f_i,g_i,f_j,g_j\}$ is an M-cocircuit for $i\neq j$;*
(c) *$\{f_i,g_i,f_j,g_j\}$ is an M-circuit for $i\neq j$.*

Proof. For any cocircuit $\{e, f, g\}$, suppose there exists $h \neq g$ such that $\{e, f, h\}$ is a cocircuit. Then $\{f, g, h\}$, $\{e, g, h\}$ are cocircuits by well-known circuit axioms [8] and 3-connectivity. Fix $f_1 \in E - e$ and let $A_1 = \{f_1\} \cup \{g : \{e, f_1, g\}$ is a corcircuit$\}$. If $A_1 \neq E - e$, choose $f_2 \in (E - e) \setminus A_1$ and let $A_2 = \{f_2\} \cup \{g : \{e, f_2, g\}$ is a cocircuit$\}$. In this way we obtain a partition $\{A_1, A_2, \ldots, A_k\}$ of $E - e$ such that each 3-element subset of $A_i + e$ is a cocircuit for each i. It follows from the fact that circuits and cocircuits cannot have exactly one element in common that each circuit not containing e must contain or be disjoint from each A_i, and that each circuit containing e must meet each A_i and must contain each A_i having $|A_i| > 2$. Fix $j \in \{1, 2, \ldots, k\}$ and choose $f_j, g_j \in A_j$. By Lemma 3.2 there exists a circuit $C \subseteq E \setminus \{e, f_j\}$ and a circuit C' such that $\{e, f_j\} \subseteq C' \subseteq C \cup \{e, f_j\}$. Thus C meets A_i for each $i \neq j$ and does not meet A_j, so $C = \cup \{A_i : i \neq j\}$. Moreover, $|A_j| = 2$ since C, and therefore C', cannot contain g_j. Thus we have $E = \{e, f_1, g_1, \ldots, f_k, g_k\}$ and Lemma 3.4(a) holds; moreover, Lemma 3.4(b) follows immediately from Lemma 3.4(a), the circuit axioms, and the definition of the A_i. We have also shown that $E \setminus \{e, f_i, g_i\}$ is an M-circuit for each i.

It suffices to show that $k = 3$. It is easily checked that we cannot have $k < 3$. If $k > 3$, consider $M' = M/\{f_k, g_k\}$. We can observe that the only circuits of M' not containing e have the form $E \setminus \{e, f_k, g_k, f_j, g_j\}$ for $j \neq k$. Since $\{e, f_j, g_j\}$ is an M'-cocircuit, e is a chord of no circuit in M'. Now, using the fact that $k > 3$, the existence of the above M'-circuits shows that $M' \setminus e$ is non-separable. Thus the assumption that $k > 3$ contradicts the minimality of M; it follows that $k = 3$.

Proof of Theorem 1.2. Suppose that M is non-separable and $M \setminus e$ is separable but e is a chord of a circuit C of M. Where $\{E_1, E_2\}$ is a separation of $M \setminus e$, we may assume $C \subseteq E_1$. Then $r(E_1 + e) = r(E_1)$ and $\{E_1 + e, E_2\}$ is a separation of M, a contradiction. Thus the condition that $M \setminus e$ be non-separable is necessary (even if M and e do not satisfy (2)).

Now suppose that M' and $M' \setminus e$ are non-separable but e is a chord of no circuit of M'. Then M' has a minor M satisfying the conditions of Lemma 3.4. Since $C = \{f_1, g_1, f_2, g_2\}$ is both a circuit and the complement of a cocircuit, therefore $3 = r(C) = r(E) - 1$ so M has rank 4. No circuit can have cardinality 3 or less so determining M is a matter of determining the 4-element circuits which contain e. It is straightforward to check that the only possibilities are that M be isomorphic to one of F_1, F_2, F_3, F_4, F_5. The proof is complete.

4. Disjoint paths

We will prove the following result, using Theorem 1.2. Note that, without the restriction (1), it fails for each F_i.

Theorem 4.1. *Let e be a cell of matroid M on E, such that M and e satisfy* (1).

There exists a set $A \subseteq E - e$ such that A spans e and such that each minimal non-empty separator of $M \setminus (E \setminus A)$ is a circuit if and only if e is contained in no cocircuit of cardinality less than 3.

Let us say that a set $P \subseteq E - e$ is an ***e-path*** of M if $P + e$ is an M-circuit; similarly $C \subseteq E - e$ is an ***e-cut*** if $C + e$ is an M-cocircuit. Since a circuit and a cocircuit cannot have exactly one element in common, it follows that if there exist as many as k mutually disjoint e-paths, then every e-cut has cardinality at least k. If M is binary, a set such as A in (4.1) provides a pair of disjoint e-paths (by the symmetric difference property).

Let $\mathcal{F}$ denote the class of binary matroids having no minor isomorphic to F_1. Then we have the following consequence of Theorem 4.1.

Corollary 4.2. *Let e be a cell of a matroid M on E such that $M \in \mathcal{F}$. There exist* 2 *(mutually) disjoint e-paths if and only if every e-cut has cardinality at least* 2.

The more general statement obtained from Corollary 4.2 when "2" is replaced by an arbitrary positive integer k, which was conjectured in an earlier version of this paper, has been proved by Seymour [11]. (His work and the present work were done independently.) In fact, by strengthening the disjoint path property, Seymour has obtained a characterization of the members of $\mathcal{F}$. Where $u = (u_j : j \in E - e)$ is a vector of non-negative integers, an (*integral*) *feasible e-flow* is a non-negative (integer-valued) vector $x = (x_P : P$ an e-path) satisfying $\sum(x_P : j \in P) \leq u_j$, for $j \in E - e$. Where each u_j is 1, an integral feasible e-flow is essentially a collection of mutually disjoint e-paths. The maximum *amount* $\sum(x_P : P$ an e-path) of an integral feasible e-flow x cannot exceed the minimum *capacity* $\sum(u_j : j \in C)$ of an e-cut C. It is proved in [11] that equality holds for every u if and only if $M \in \mathcal{F}$. As well as being an extensive generalization of Corollary 4.2, this result includes the max flow-min cut theorem for regular matroids, due to Minty [10, 6].

A weakening of the notion of integral feasible flow is obtained by dropping the integrality requirement. Again, the amount of a feasible flow is bounded above by the capacity of a cut. It is a consequence of Seymour's result that, if $M \in \mathcal{F}$, there exists a feasible e-flow of integer amount k if and only if there exists an integral e-flow of amount k. However, this is not true in general; where $u_j = 1$, $j \in E - e$, we can put $x_P = 1/2$ for each e-path P of F_1 and obtain a feasible e-flow of amount 2. More generally, it is not difficult to show that for any u the weakened max flow-min cut theorem holds for F_1. Thus determining the matroids having this property for any u is a different problem from the one solved in [11]. Binary matroids not having this property have been found by Bixby [3] and Seymour [12]. Seymour has given three different examples, one of which is equivalent to Bixby's. Notice that M and e have the weak max flow-min cut property for every u if and only if the polyhedron $\{y = (y_j : j \in E - e): y \geq 0;\ \sum(y_j : j \in P) \geq 1,\ P$ an e-path$\}$ has as vertices precisely the incidence vectors of e-cuts. (These remarks are closely related to the paper [7] of Fulkerson.)

Proof of Theorem 4.1. We prove first that the condition is necessary for the existence of a set A as in Theorem 4.1. Suppose $A \subseteq E - e$ has the properties desired. Then e cannot be an isthmus, since A would not span e. If $\{e, f\}$ is a cocircuit for some $f \in E - e$, then $f \in A$, since otherwise a circuit would contain e and not f. But f will be an isthmus of $M \setminus (E \setminus A)$ because $\{e, f\}$ is a cocircuit of M. Thus not every minimal non-empty separator of this matroid is a circuit. This proves necessity.

It is easy to see that the condition is sufficient if e is a loop. Otherwise, we may assume M is non-separable, for if A has the required property and E_1 is the minimal non-empty separator of M containing e, then $E_1 \cap A$ has the required properties. Finally the result is easily checked for matroids having fewer than three cells. If $M \setminus e$ is non-separable, the result follows from Theorem 1.2. Otherwise, let $S_1, S_2, \ldots, S_k$ be the minimal non-empty separators of $M \setminus e$, where $k \geq 2$. Each $|S_i| \geq 2$, since otherwise $M \setminus e$ would have an isthmus and so M would have a 2-element cocircuit containing e. Thus $r(S_i) + r(E \setminus S_i) \leq r(E) + 1$ for each i; by (3) there exist matroids M_i on $S_i + f_i$ and M'_i on $(E \setminus S_i) + f_i$ such that $M = M_i * M'_i$ for each i. Now (1) is satisfied when M is replaced by M_i and e is replaced by f_i, by (4) and the fact that M and e satisfy (1). Thus by Theorem 1.2, there exists for each i, an M_i-circuit C_i of which f_i is a chord in M_i. We will show that $A = \bigcup \{C_i : i = 1, 2, \ldots, k\}$ has the properties required.

Since each C_i is a circuit of M_i not containing f_i, therefore each C_i is a circuit of M. Thus the C_i are the minimal non-empty separators of $M \setminus (E \setminus A)$. Therefore we need only prove that A M-spans e. Since M is non-separable and has at least 3 cells, there is an M-circuit C containing e. If $C \subseteq A + e$, we are done; otherwise suppose C meets $S_j \setminus C_j$ for some $j \in \{1, \ldots, k\}$. Since f_j is a chord of C_j in M_j, there exists an M-circuit D_j with $f_j \in D_j \subseteq C_j + f_j$. Then by (2) $C' = (D_j - f_j) \cup (C \setminus S_j)$ is an M-circuit, not meeting $S_j \setminus C_j$. We replace C by C' and continue the process until we find an M-circuit C with $e \in C \subseteq A + e$. Thus A M-spans e, and the proof is complete.

Finally, we mention one other problem related to disjoint paths. Let us say that e-paths P_1, P_2 of M are *strongly disjoint* if they are disjoint and $P_1 \cup P_2$ is an M-circuit. For a graphic matroid a set of mutually strongly disjoint e-paths corresponds to a set of mutually vertex-disjoint paths in $G - e$ joining the ends of e. Using the relationship between matroid connectivity and graph connectivity, we obtain the following restatement of Menger's theorem.

Theorem 4.3. *Let k be an integer, $k \geq 2$, and let e be a cell of the graphic matroid M, such that e is contained in no 1-element or 2-element circuit. There exist k mutually strongly disjoint e-paths if and only if there does not exist a partition $\{E_1, E_2\}$ of $E - e$ such that $r(E_1) + r(E_2) < r(E - e) + k - 1$ and neither E_1 nor E_2 M-spans e.*

As usual, it is easily checked that the condition of (4.3) is necessary for the existence of k mutually strongly disjoint e-paths in any matroid. It is natural to ask whether Theorem 4.3 can be extended to a larger class of binary matroids than the graphic matroids. For the case $k = 2$ it can be shown that Theorem 4.3 is true for $M \in \mathscr{F}$; this is a straightforward consequence of Theorem 1.2. However, for $k = 3$, Theorem 4.3 does not hold even for regular matroids; it can be checked that the dual of the graphic matroid obtained from the bicomplete graph $K_{3,3}$ is a counterexample.

5. Matroid sums

All matroids in this section are assumed to have no loop. A matroid is *sum-prime* if it cannot be expressed as the sum of 2 matroids, each having rank at least 1. Apart from transversal matroid theory, which may be regarded as the study of matroids which are sums of rank-one matroids, very little seems to be known about what kinds of sum decompositions a matroid can have. The only result characterizing sum-prime matroids is a result of Lovász and Recski [9] on graphic matroids. Their result is the special case of Theorem 5.1 below where M is graphic.

Theorem 5.1. *A matroid $M \in \mathscr{F}$ is sum-prime if and only if M is non-separable and $M \setminus e$ is non-separable for each cell e of M.*

Our proof of Theorem 5.1 follows the proof in [9], except that Theorem 1.2 is used in place of Theorem 1.1; in fact, Theorem 1.2 was discovered in an attempt to generalize the result on graphic sum-prime matroids to a larger class of binary matroids. A *line* in a matroid M is a set L which is the disjoint union of sets $S_1, S_2, \ldots, S_k$ such that, for each i, $L \setminus S_i$ is an M-circuit. Using a well-known formula for the rank function of the sum M of matroids M_1 and M_2 on E_1 and E_2 having rank functions r_1 and r_2, it is shown in [9], for any line L of M as above and $i \neq j$, that

$$r_1(E_1 \cap (L \setminus (S_i \cup S_j))) + r_2(E_2 \cap (L \setminus (S_i \cup S_j))) = |L \setminus (S_i \cup S_j)|.$$

Proof of Theorem 5.1. If M is separable, it is clearly not prime. If it is non-separable and $M \setminus e$ has a separation $\{E_1, E_2\}$ for some cell e of M, we let $M_1 = M/E_2$ and $M_2 = M/E_1$. It is clear that the circuits of $M_1 + M_2$ not containing e are precisely the circuits of M not containing e. The circuits of $M_1 + M_2$ containing e are precisely unions $C_1 \cup C_2$ of M_1-circuits C_1 containing e with M_2-circuits C_2 containing e. Thus to show that $M = M_1 + M_2$, it is enough to show that M_i-circuits containing e are precisely sets of the form $(C \cap E_i) + e$ for M-circuits C containing e. But this is easy to see.

Now suppose that M is non-separable and that $M \setminus e$ is non-separable for each cell e of E. Then each such e is a chord of a circuit C of M. Then $C + e$ is a line with corresponding partition $S_1 = \{e\}, S_2, S_3$ since M is binary. Thus, if $M = M_1 + M_2$, then $r_1(\{e\} \cap E_1) + r_2(\{e\} \cap E_2) = 1$, by the result of Lovász and Recski. Thus e is an element of E_1 or E_2, but not both. But then $\{E_1, E_2\}$ is a separation of M, a contradiction. The proof is finished.

Since the matroid F_1 can be shown to be sum-prime, it is likely that the above result extends to the class of all binary matroids. However, the present method of proof clearly does not. We also observe that there exists a good matroid algorithm for deciding whether a member of $\mathcal{F}$ is sum-prime; this follows from Theorem 5.1 and an algorithm for testing for separability given in [4].

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